

Light-cone analysis of ungauged and topologically gauged BLG theories

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ABSTRACT: We consider three-dimensional maximally superconformal Bagger-Lambert-Gustavsson (BLG) theory and its topologically gauged version (constructed recently in [1]) in the light-cone gauge. After eliminating the entire Chern-Simons gauge field, the ungauged BLG light-cone theory looks more conventional and, apart from the order of the interaction terms, resembles $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. The light-cone superspace version of the BLG theory is given at quadratic order together with a suggested form for the quartic terms. Some problems with constructing the sixth order interaction terms are also discussed. In the topologically gauged case, we analyze the field equations related to the three Chern-Simons type terms of $\mathcal{N} = 8$ conformal supergravity and discuss some of the special features of this theory and its couplings to BLG.

KEYWORDS: String theory, M-theory, Branes.

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1. Introduction

A three-dimensional maximally superconformal Chern-Simons theory was recently constructed in [2, 3, 4]. This Bagger-Lambert-Gustavsson (BLG) theory was originally proposed to describe multiple M2-branes an interpretation that soon, however, met with a number of problems stemming from the algebraic structure of the theory. The Lagrangian contains a four-indexed structure constant defining a three-algebra which is known to have essentially only one (finite-dimensional) realization, A_4 , related to the Lie algebra $so(4)$ [5, 6]. This is limiting the role of this theory to stacks of two M2-branes.

To describe stacks of more than two M2 branes there are basically two different options discussed in the literature. By relaxing the assumption of a positive definite metric on the algebra any Lie algebra can be accommodated. However, a degenerate metric [7] leads to field equations which cannot be integrated to a Lagrangian unless the potentially dangerous modes can be rendered harmless. Postulating that these are constant makes a Lagrangian possible but also seems to alter the theory in a non-trivial way. Similar conclusions probably apply also to the case of Lorentzian metrics.

The second possibility to avoid the A_4 uniqueness result mentioned above is to reduce the number of (manifest) supersymmetries from the maximal $\mathcal{N} = 8$ to $\mathcal{N} = 6$ as done in the work of Aharony, Bergman, Jafferis and Maldacena (ABJM) [8]. These theories are, e.g., known to exist for gauge groups $U(N) \times U(N)$ where N can be any positive integer.

In four dimensions the corresponding maximally supersymmetric theory (with 16 supercharges) is the $\mathcal{N} = 4$ super-Yang-Mills theory. This is a conventional theory with standard kinetic terms for all fields, and with interaction terms dictated by the conformal invariance. Instead, in three dimensions the BLG superconformal theory with 16 supercharges, now corresponding to $\mathcal{N} = 8$, has several unusual features. One is the appearance of three-algebras (and perhaps generalized Jordan triple systems [9]¹) and their associated structure constants that satisfy the so called fundamental identity. A second distinguishing feature is the fact that the BLG gauge potential has dynamics governed by a Chern-Simons term [11, 2, 3, 4, 12] in the action, and is therefore not an independent field on-shell. A standard kinetic Yang-Mills term is of course in conflict with conformal invariance in three dimensions and will not be part of this theory. However, such non-conformal Yang-Mills theories will in general flow to an infrared fix-point where interacting superconformal theories become relevant.

It is interesting to note that if written in the light-cone gauge these two theories, $\mathcal{N} = 8$ BLG in three and $\mathcal{N} = 4$ SYM in four dimensions [13, 14, 15], tend to look a bit more similar, but with the crucial difference that the power of the interaction terms differ. (As we will see later there are some differences also in the way the ∂_-

¹See also [10] for the special properties of the $\mathcal{N} = 6$ structure constants used in this context.

derivatives appear.) One might still hope, however, that after collecting all fields in these two theories into superfields, one will find that the Lagrangians will share the essential features needed to conclude that their quantum properties coincide, i.e., that the BLG theory is ultra-violet finite to all orders in perturbation theory for the same reasons as in the $\mathcal{N} = 4$ SYM case [13, 14, 15]. One purpose of this note is to start the development of such a light-cone superspace formulation of superconformal theories in three dimensions.

We will also use light-cone methods to investigate the physical properties (degrees of freedom) of topologically gauged BLG (i.e., $\mathcal{N} = 8$ superconformal supergravity coupled to BLG) recently discussed in [1]. This supergravity theory consists of one ordinary Chern-Simons term (containing one derivative) related to the R-symmetry and two slightly unusual higher derivative Chern-Simons terms for the spin connection (three derivatives) and the Rarita-Schwinger field (two derivatives) [16, 17]. The physical mode content of these latter two terms are therefore less obvious but can be easily analyzed in the light-cone gauge as will be demonstrated below.

This note is organized as follows. In section 2 we review the ordinary ungauged BLG theory and discuss some of its light-cone properties. While the complete light-cone theory is presented in component form, the superspace formulation is constructed only at quadratic order. We do also suggest a possible answer at quartic order. However, lacking the answer in light-cone superspace for the sixth order interactions means that a proof of finiteness using these methods will be out of reach for the moment. The complications that are the reason for this are discussed briefly. Section three starts with a short review of the $\mathcal{N} = 8$ superconformal supergravity theory followed by a light-cone analysis of its degrees of freedom and couplings to BLG matter. Section four contains some additional comments and conclusions.

2. The BLG theory

This section contains a review of the BLG theory followed by a derivation of its light-cone Lagrangian. We also discuss some issues that arise when trying to rewrite this theory in $\mathcal{N} = 8$ light-cone superspace.

2.1 Review of the BLG theory

The BLG theory contains three different fields, the two propagating ones X_A^I and Ψ_A , which are scalars and spinors, respectively, on the M2-brane, and the auxiliary gauge field $\tilde{A}_{\mu B}^A$. Here the indices A,B,.. are connected to the three-algebra and a basis T^A , while the I,J,K,.. are $so(8)$ vector indices; in addition the spinors transform under a spinor representation of $so(8)$ but the corresponding index is not written explicitly. Indices μ, ν .., etc, are vector indices on the flat M2-brane world volume.

For these fields one can write down supersymmetry transformation rules and covariant field equations. This does not require a metric on the three algebra which

means that the structure constants should be written as $f^{ABC}{}_D$ and consequently the *fundamental identity* reads

$$f^{ABC}{}_G f^{EFG}{}_D = 3f^{EF[A}{}_G f^{BC]G}{}_D, \quad (2.1)$$

which has as an alternate form [7]

$$f^{[ABC}{}_G f^{E]FG}{}_D = 0. \quad (2.2)$$

The construction of a Lagrangian forces us to introduce a metric on the three algebra, and if one wants to describe more general Lie algebras than $so(4)$, this metric must be generate [7] or indefinite [18, 19, 20, 21] which, however, may just constitute a reformulation of D2-brane systems. We also need to introduce the gauge field $A_{\mu AB}$ which is related to the previously defined one as follows

$$\tilde{A}_{\mu B}^A = A_{\mu CD} f^{CDA}{}_B. \quad (2.3)$$

We will not need to concern ourselves in this paper with questions related to the exact form of the structure constants as long as they are compatible with the existence of a Lagrangian. The BLG Lagrangian is [4]

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(D_\mu X^A)(D^\mu X_A) + \frac{i}{2}\bar{\Psi}^A \Gamma^\mu D_\mu \Psi_A + \frac{i}{4}\bar{\Psi}_B \Gamma_{IJ} X_C^I X_D^J \Psi_A f^{ABCD} \\ & -V + \frac{1}{2}\varepsilon^{\mu\nu\lambda} \left(f^{ABCD} A_{\mu AB} \partial_\nu A_{\lambda CD} + \frac{2}{3} f^{CDA}{}_G f^{EFGB} A_{\mu AB} A_{\nu CD} A_{\lambda EF} \right), \end{aligned} \quad (2.4)$$

where

$$V = \frac{1}{12} f^{ABCD} f^{EFG}{}_D X_A^I X_B^J X_C^K X_E^I X_F^J X_G^K. \quad (2.5)$$

In order to rewrite this action in the light-cone gauge we need to be able to express all non-propagating fields in terms of propagating ones. These expressions are obtained from the components of the field equations that become algebraic once the possibility to divide by ∂_- becomes available. This is one of the crucial features of the light-cone gauge. The field equations derived from the action above are

$$\begin{aligned} 0 &= \Gamma^\mu D_\mu \Psi_A + \frac{1}{2} \Gamma_{IJ} X_C^I X_D^J \Psi_B f^{CDB}{}_A, \\ 0 &= D^2 X_A^I - \frac{i}{2} \bar{\Psi}_C \Gamma^I{}_J X_D^J \Psi_B f^{CDB}{}_A + \frac{1}{2} f^{BCD}{}_A f^{EFG}{}_D X_B^J X_C^K X_E^I X_F^J X_G^K, \\ 0 &= \tilde{F}_{\mu\nu}{}^B{}_A - \varepsilon_{\mu\nu\lambda} (X_C^J D^\lambda X_D^J + \frac{i}{2} \bar{\Psi}_C \Gamma^\lambda \Psi_D) f^{CDB}{}_A, \end{aligned} \quad (2.6)$$

where the covariant derivative and field strength are defined by

$$\begin{aligned} (D_\mu X)_A &= \partial_\mu X_A - \tilde{A}_\mu{}^B{}_A X_B, \\ \tilde{F}_{\mu\nu}{}^B{}_A &= 2 \left(\partial_{[\mu} \tilde{A}_{\nu]}{}^B{}_A + \tilde{A}_{[\mu}{}^B{}_{|C|} \tilde{A}_{\nu]}{}^C{}_A \right). \end{aligned} \quad (2.7)$$

We will also need the supersymmetry transformation rules in the analysis below. They are

$$\begin{aligned}\delta X_A^I &= i\bar{\epsilon}\Gamma^I\Psi_A, \\ \delta\Psi_A &= D_\mu X_A^I\gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X_B^IX_C^JX_D^K\Gamma^{IJK}\epsilon f^{BCD}{}_A, \\ \delta\tilde{A}_\mu{}^A{}_B &= i\bar{\epsilon}\gamma_\mu\Gamma^IX_C^I\psi_D f^{CDA}{}_B.\end{aligned}\tag{2.8}$$

2.2 The BLG action in the light-cone gauge

The light-cone quantities used in this paper are all defined in accordance with the light-cone coordinates

$$x^+ := 1/\sqrt{2}(x^0 + x^1), \quad x^- := 1/\sqrt{2}(x^0 - x^1), \quad x := x^2,\tag{2.9}$$

which means that the Lorentzian scalar product is

$$-x^0y^0 + x^1y^1 + x^2y^2 = -x^+y^- - x^-y^+ + xy.\tag{2.10}$$

We choose the light-cone gauge on the vector field $A_\mu = (A_-, A_+, A)$ as

$$A_- = 0.\tag{2.11}$$

Implementing these definitions and the gauge condition in the bosonic part of the BLG action, it becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}X_A^I(-2\partial_-\partial_+ + \partial^2)X_A^I - X_A^I\tilde{A}_+^{AB}\partial_-X_B^I + X_A^I\tilde{A}^{AB}\partial X_B^I + \frac{1}{2}X_A^I(\tilde{A}^2)^{AB}X_B^I \\ &\quad - A_+^{AB}\partial_-\tilde{A}_{AB} - V,\end{aligned}\tag{2.12}$$

where one may note that the interactions in the Chern-Simons term are being put to zero by the gauge choice. The next step is to solve the field equations involving $F_{\mu\nu}$ and insert the solutions back into the above expression. We find that the F_{-+} component gives

$$\tilde{A}_+^{AB} = -\frac{1}{\partial_-}(X_C^I\partial X_D^I - X_C^IX_E^I\tilde{A}^E{}_D + \frac{i}{2}\bar{\Psi}_C\gamma\Psi_D)f^{CDAB},\tag{2.13}$$

while F_{-2} implies that (we drop the index 2 in the following)

$$\tilde{A}^{AB} = \frac{1}{\partial_-}(X_C^I\partial_-X_D^I + \frac{i}{2}\bar{\Psi}_C\gamma_-\Psi_D)f^{CDAB}.\tag{2.14}$$

The equation involving the remaining component of the field strength, F_{+2} , then becomes an identity which may also be seen from the Bianchi identity by using it to solve for this last component in terms of the other two. Here we emphasize that ∂_- always acts just on the field following directly after it, while $\frac{1}{\partial_-}$ acts on the whole expression in the parenthesis following it.

We see here that to express the two non-zero components of A_μ entirely in terms of matter fields we must also insert A back into A_+ . This is not needed in the analysis of four-dimensional $\mathcal{N} = 4$ SYM since in that case A represents independent degrees of freedom. In the analysis of the M2 system performed here, eliminating also A is straightforward and will, e.g., not cause any problems related to functional determinants arising when doing this in a path integral. This will be shown explicitly below. However, to rewrite the full Lagrangian in light-cone superspace is complicated and we will present the result only at quadratic order and suggest a possible answer at quartic order. This problem and some related issues will be discussed in the next subsection.

To repeat this for the spinor field Ψ , we start from its field equation given in the previous subsection and solve for half the spinor. This is done as follows: split Ψ into $\Psi^{(+)} = P^+\Psi = \frac{1}{2}(1 + \gamma^2)\Psi$ and $\Psi^{(-)} = P^-\Psi = \frac{1}{2}(1 - \gamma^2)\Psi$ and use the fact that the field equation produces the two equations

$$-\gamma^2 D_2 \gamma^- \Psi_A^{(+)} + \gamma^- \gamma^+ D_+ \Psi_A^{(+)} + \frac{1}{2} \Gamma_{IJ} \gamma^- \Psi_B^{(+)} X_C^I X_D^J f^{CDB}{}_A = 0, \quad (2.15)$$

and

$$-\gamma^2 D_2 \gamma^+ \Psi_A^{(-)} + \gamma^+ \gamma^- D_- \Psi_A^{(-)} + \frac{1}{2} \Gamma_{IJ} \gamma^+ \Psi_B^{(-)} X_C^I X_D^J f^{CDB}{}_A = 0, \quad (2.16)$$

to solve for $\Psi^{(+)}$. We find (from the latter equation suppressing the index 2)

$$\Psi_A^{(+)} = -\frac{1}{2} \partial_-^{-1} (\gamma \partial \gamma^+ \Psi_A^{(-)} + \gamma \tilde{A}_A{}^B \gamma^+ \Psi_B^{(-)}) + \frac{1}{4} \partial_-^{-1} \Gamma_{IJ} \gamma^+ \Psi_B^{(-)} X_C^I X_D^J f^{CDB}{}_A. \quad (2.17)$$

Also in this expression we need to insert \tilde{A}^{AB} given in terms of the matter fields above. As seen explicitly below, this will produce terms in the light-cone Lagrangian that are sixth order in fermionic fields.

We will now show in detail how to use the path integral to integrate out all the dependent degrees of freedom, i.e., the remaining two components of the Chern-Simons vector field and half of the BLG spinor. From the form of the bosonic part of the action in the light-cone gauge presented above, we see that the path integral is (suppressing the R-symmetry indices and leaving out the fermions for the moment to simplify the argument)

$$Z[X] = \int \mathcal{D}[A, A_+] \exp i \int d^3x (-A_+^{AB} \partial_- \tilde{A}^{AB} - \frac{1}{2} \tilde{A}^{AB} (X^{[A} X_{C]} \delta_{D]}^B \tilde{A}^{CD} + \tilde{A}^{AB} (X_A \partial X_B) - \tilde{A}_+^{AB} (X_A \partial_- X_B)), \quad (2.18)$$

where the parenthesis (XX) indicates a scalar product in the R-symmetry vector indices. This can be written as

$$Z[X] = \int \mathcal{D}[\mathcal{A}] \exp i \int d^3x (\frac{1}{2} \mathcal{A}^T M \mathcal{A} + \mathcal{A}^T \mathcal{J}) \quad (2.19)$$

where, if we use the definitions

$$\mathcal{A}^T = (\hat{A}, \tilde{A}) = (\partial_- A_+, \tilde{A}) \quad (2.20)$$

and

$$(\mathcal{J}^T)^{AB} = (J^{AB}, \tilde{J}^{AB}) = (f^{ABCD} \frac{1}{\partial_-} (X_C \partial_- X_D), (X^{[A} \partial X^{B]})), \quad (2.21)$$

we find that the matrix M is

$$M_{AB}{}^{CD} = \begin{pmatrix} 0 & \delta_{AB}^{CD} \\ \delta_{AB}^{CD} & H_{AB}{}^{CD} \end{pmatrix}, \quad (2.22)$$

with H given by

$$H_{AB}{}^{CD} = -\delta_{[A}^{[C} (X_{B]} X^{D]}. \quad (2.23)$$

Performing the path integral over the vector field components A, A_+ gives a functional determinant that is just $\det(\partial_-)^{-\frac{1}{2}}$ and hence field independent. Furthermore, the matrix M will appear in the path integral through its inverse

$$(M^{-1})_{AB}{}^{CD} = \begin{pmatrix} -H_{AB}{}^{CD} & \delta_{AB}^{CD} \\ \delta_{AB}^{CD} & 0 \end{pmatrix}. \quad (2.24)$$

Explicitly we find (neglecting the determinant)

$$Z[X] = \exp i \int d^3x \left(-\frac{1}{2} \mathcal{J}^T M^{-1} \mathcal{J} \right), \quad (2.25)$$

where the integrand in the exponent reads

$$\begin{aligned} \frac{1}{2} \mathcal{J}^T M^{-1} \mathcal{J} &= \frac{1}{2} f^{ABCD} \frac{1}{\partial_-} (X_C \partial_- X_D) (X_{[A} X^{B]} \delta_{B]}^{B'}) f^{A'B'C'D'} \frac{1}{\partial_-} (X_{C'} \partial_- X_{D'}) \\ &+ f^{ABCD} (X_A \partial X_B) \frac{1}{\partial_-} (X_C \partial_- X_D). \end{aligned} \quad (2.26)$$

This result is of course the same as that obtained by solving the field equations for the vector field and inserting it back into the action.

Adding the fermionic Lagrangian²

$$L_{fermion} = \frac{i}{2} (\sqrt{2} \Psi^+ \partial_- \Psi^+ + 2 \Psi^+ (\partial + \tilde{A}) \Psi^- + \sqrt{2} \Psi^- (\partial_+ + \tilde{A}_+) \Psi^-) \quad (2.27)$$

to the above discussion just changes the matrix M to

$$\hat{M}_{AB}{}^{CD} = \begin{pmatrix} 0 & \delta_{AB}^{CD} & 0 \\ \delta_{AB}^{CD} & H_{AB}{}^{CD} & -i \delta_{[A}^C \Psi_{B]}^{(-)} \\ 0 & i \delta_A^{[C} \Psi^{D]} (-) & i \sqrt{2} \delta_A^C \partial_- \end{pmatrix}, \quad (2.28)$$

²We are working with the following conventions: $\gamma^0 = i\sigma^2$, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^3$ and the Dirac conjugate is $\bar{\Psi} = \Psi^\dagger (-\gamma^0)$.

with the same H as before. Note that even after including the fermions this matrix has a nice inverse

$$(\hat{M}^{-1})_{AB}{}^{CD} = \begin{pmatrix} -\hat{H}_{AB}{}^{CD} & \delta_{AB}^{CD} & -\frac{1}{\sqrt{2}}\delta_{[A}^C\Psi_{B]}^{(-)}\frac{1}{\partial_-} \\ \delta_{AB}^{CD} & 0 & 0 \\ \frac{1}{\sqrt{2}}\frac{1}{\partial_-}\delta_A^{[C}\Psi^{D](-)} & 0 & -\frac{i}{\sqrt{2}}\delta_A^C\frac{1}{\partial_-} \end{pmatrix}, \quad (2.29)$$

where \hat{H} is given below. It should be noted here, however, that this matrix is a slightly more delicate operator than \hat{M} since the inverse derivatives $\frac{1}{\partial_-}$ are defined to act on everything to the right of it (in particular the current \mathcal{J}).

To obtain these matrices we have defined the fermion extended quantities,

$$\hat{\mathcal{A}} = (\partial_- A_+^{AB}, \tilde{A}^{AB}, \Psi_A^{(+)}), \quad (2.30)$$

and the current

$$\hat{\mathcal{J}} = (\hat{J}^{AB}, \tilde{J}^{AB}, J_A), \quad (2.31)$$

where the respective components are

$$\hat{J}^{AB} = f^{ABCD}\frac{1}{\partial_-}(X_C\partial_-X_D) - f^{ABCD}\frac{i}{\sqrt{2}}\frac{1}{\partial_-}(\Psi_C^{(-)}\Psi_D^{(-)}), \quad (2.32)$$

$$\tilde{J}^{AB} = X^{[A}\partial_-X^{B]}, \quad (2.33)$$

$$J_A = i(\partial\Psi_A^{(-)}) + \frac{i}{2}\Gamma_{IJ}\Psi_B^{(-)}X_C^IX_D^Jf_A{}^{BCD}. \quad (2.34)$$

Performing the $\Psi^{(+)}$ part of the path integral is equivalent to replacing H with its fermion corrected version

$$\hat{H}_{AB}{}^{CD} = -\delta_{[A}^{[C}(X_{B]}X^{D]}) + \frac{i}{\sqrt{2}}\delta_{[A}^{[C}\Psi_{B]}^{(-)}\frac{1}{\partial_-}\Psi^{D](-)}. \quad (2.35)$$

This can be seen by considering the superdeterminant

$$s\det\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det^{-1}(B)\det(A - CB^{-1}D) \quad (2.36)$$

where the expression in the second determinant corresponds to the new \hat{H} .

Collecting the above results, the action takes the following form when expressed in terms of only propagating degrees of freedom:

$$\mathcal{L} = \frac{1}{2}X_A^I\Box X_A^I + \frac{i}{\sqrt{2}}\Psi_A^{(-)}\partial_+\Psi_A^{(-)} - \frac{1}{2}\hat{\mathcal{J}}^T\hat{M}^{-1}\hat{\mathcal{J}} - V, \quad (2.37)$$

where

$$\begin{aligned} \frac{1}{2}\hat{\mathcal{J}}^\dagger\hat{M}^{-1}\hat{\mathcal{J}} &= -\frac{1}{2}\hat{J}_{AB}^\dagger\hat{H}^{AB,CD}\hat{J}_{CD} + \frac{i}{2\sqrt{2}}J_A^\dagger\frac{1}{\partial_-}J^A \\ &\quad + \hat{J}_{AB}^\dagger\tilde{J}^{AB} + \frac{1}{\sqrt{2}}\hat{J}_{AB}^\dagger\Psi_B^{(-)}(\frac{1}{\partial_-}J_A). \end{aligned} \quad (2.38)$$

These expressions have in some cases a more complicated $\frac{1}{\partial_-}$ structure than is known from, e.g., SYM in four dimensions. To see this let us consider the first one of these terms

$$\begin{aligned}
& -\frac{1}{2}\hat{J}_{AB}^\dagger \hat{H}^{AB,CD} \hat{J}_{CD} = \frac{1}{2}f^{ABCD}f_A^{FGH} \times \\
& \quad \left(\frac{1}{\partial_-}(X_C\partial_-X_D - \frac{i}{\sqrt{2}}\Psi_C^{(-)}\Psi_D^{(-)})(X_BX_F)\frac{1}{\partial_-}(X_G\partial_-X_H - \frac{i}{\sqrt{2}}\Psi_G^{(-)}\Psi_H^{(-)})\right. \\
& \quad \left.- \frac{i}{\sqrt{2}}\frac{1}{\partial_-}(X_C\partial_-X_D - \frac{i}{\sqrt{2}}\Psi_C^{(-)}\Psi_D^{(-)})\Psi_B^{(-)}\frac{1}{\partial_-}(\Psi_F^{(-)}\frac{1}{\partial_-}(X_G\partial_-X_H - \frac{i}{\sqrt{2}}\Psi_G^{(-)}\Psi_H^{(-)}))\right),
\end{aligned} \tag{2.39}$$

where the last term contains nested inverse derivatives instead of simpler combinations like $\frac{1}{\partial_-^2}$. Despite the intricate structure of these sixth order terms they must be the ones that are required in order to promote the original X^6 term in the BLG Lagrangian to light-cone superspace. One of the terms above is purely fermionic with three inverse ∂_- derivatives to give it the correct dimension.

Note also that the fermionic kinetic term in the light-cone Lagrangian should be $-\frac{i}{2\sqrt{2}}\Psi_A^{(-)}\frac{\square}{\partial_-}\Psi_A^{(-)}$ and that the missing piece comes from the second term on the right hand side of (2.38).

Finally we remark that to fit the fields into a superfield, as further discussed in the next subsection, we need to decompose X^I and $\Psi^{(-)}$ into representations of $SU(4)$ according to $\frac{1}{2}X^IY^I = \frac{1}{2}(A\bar{B} + \bar{A}B) + \frac{1}{4}C_{mn}\bar{D}^{mn}$ and $\Psi^{(-)}\Psi'^{(-)} = \chi_m\bar{\chi}'^m + \bar{\chi}^m\chi'_m$. It is then a trivial exercise to write out the whole action in terms of the $8+8$ independent light-cone degrees of freedom A, C_{mn}, χ_m .

2.3 Light-cone superspace

The purpose of this subsection is to try to organize the propagating degrees of freedom in such a way that they will all fit into one single light-cone superfield. The prototype superfield is the one previously used in the context of four-dimensional $\mathcal{N} = 4$ super-Yang-Mills [13, 14, 15] to prove the all loop finiteness of that theory (which ultimately is one of the goals also here³). As already discussed briefly at the end of the previous subsection, by breaking $SO(8)$ to $SU(4) \times U(1)$ we can define an $SU(4)$ complex scalar A and a complex field C_{mn} , with $m, n, ..$ each a 4 of $SU(4)$. The latter field, being antisymmetric and selfdual in the indices, is then transforming in a 6 of $SU(4)$. Explicitly we define (suppressing the three-algebra indices)

$$\frac{1}{2}X^IX^I = A\bar{A} + \frac{1}{4}C_{mn}\bar{C}^{mn}, \quad \Psi\Psi' = \bar{\chi}^m\chi'_m + \chi_m\bar{\chi}'^m, \tag{2.40}$$

where the spinor bilinear $\Psi\Psi$ is the scalar product in their $SO(8)$ spinor indices. Note that A need not be confused with components of the gauge field since they are

³See, e.g., [22, 23] for some results on the renormalization properties of Chern-Simons theories in three dimensions. The actual BLG theory is discussed at one loop in [24].

all eliminated at this stage. One may now use these expressions, together with the decomposition in the action and write it completely in terms of light-cone variables. This should be a suitable starting point in the search for a light-cone superspace formulation of the BLG theory.

The superspace version of the action at quadratic order is

$$S_2 = -2^{-7} \int d^3x d^4\theta d^4\bar{\theta} (\Phi \frac{\square}{\partial_-^2} \bar{\Phi}). \quad (2.41)$$

We now need to define the superfield Φ (i.e., relate it to the component fields) and verify that this superspace action provides the correct component expression⁴. To do so we have to identify the covariant derivatives d_m that are related to the linearly realized generators q_m of the supersymmetry algebra: d_m is the P_- projected part of the covariant derivative D_α (here α is an $SO(2,1)$ spinor index) and satisfies, after decomposing its $SO(8)$ spinor index into 4 and $\bar{4}$,

$$\{d_m, \bar{d}^n\} = 2\sqrt{2}i\delta_m^n \partial_-. \quad (2.42)$$

Using standard superspace techniques, we replace the $d^4\theta$ in the superspace measure with the covariant derivative expression $d^4 = \frac{1}{4!}\epsilon^{mnpq}d_md_nd_pd_q$ (and similarly for its complex conjugate) and evaluate it on a generic integrand. Imposing that Φ is chiral, $\bar{d}^m\Phi = 0$, we find that

$$\begin{aligned} d^4\bar{d}^4(\Phi\bar{\Phi}) &= d^4\Phi\bar{d}^4\bar{\Phi} - \frac{1}{6}d_md_nd_p\Phi\bar{d}^m\bar{d}^n\bar{d}^p(2\sqrt{2}i\partial_-)\bar{\Phi} - \frac{1}{2}d_md_n\Phi\bar{d}^m\bar{d}^n(2\sqrt{2}i\partial_-)^2\bar{\Phi} \\ &\quad + d_m\Phi\bar{d}^m(2\sqrt{2}i\partial_-)^3\bar{\Phi} + \Phi(2\sqrt{2}i\partial_-)^4\bar{\Phi}. \end{aligned} \quad (2.43)$$

Note that so far only the supersymmetry algebra and the (anti)chirality constraint have been used which means that the superfields in this expression can be replaced by any kind of composite (anti)chiral combinations of the basic superfield.

The next step is to use this superspace expression to derive the component form of the quadratic Lagrangian. To do this we first use the duality constraint

$$d_md_n\Phi = \frac{1}{2}\epsilon_{mnpq}\bar{d}^p\bar{d}^q\bar{\Phi} \quad (2.44)$$

(implying for instance that $d^4\Phi = -(2\sqrt{2}i\partial_-)^2\bar{\Phi}$) to arrive at

$$\begin{aligned} d^4\bar{d}^4(\Phi\bar{\Phi}) &= 2\Phi(2\sqrt{2}i\partial_-)^4\bar{\Phi} - \frac{1}{2}d_md_n\Phi\bar{d}^m\bar{d}^n(2\sqrt{2}i\partial_-)^2\bar{\Phi} \\ &\quad + 2d_m\Phi\bar{d}^m(2\sqrt{2}i\partial_-)^3\bar{\Phi}, \end{aligned} \quad (2.45)$$

where we have allowed also for integration by parts. This gives the result

$$\begin{aligned} d^4\bar{d}^4(\Phi \frac{\square}{\partial_-^2} \bar{\Phi}) &= -2^7(\partial_- \Phi \square \partial_- \bar{\Phi} - \frac{1}{2^5}d_md_n\Phi \square \bar{d}^m\bar{d}^n\bar{\Phi} \\ &\quad - \frac{i}{2\sqrt{2}}\partial_- d_m\Phi \square \bar{d}^m\bar{\Phi}). \end{aligned} \quad (2.46)$$

⁴Of course, this becomes non-trivial first when discussing the interaction terms.

Finally, inserting the values of the superfield and its derivatives at $\theta = 0$, i.e.,

$$\Phi|_0 = \frac{1}{\partial_-}A, \quad d_m\Phi|_0 = \sqrt{2}\frac{1}{\partial_-}\chi_m, \quad d_md_n\Phi|_0 = 2\sqrt{2}iC_{mn}, \quad (2.47)$$

into the superspace action above gives the wanted answer

$$\mathcal{L}_2 = A\Box\bar{A} + \frac{1}{4}C_{mn}\Box\bar{C}^{mn} - \frac{i}{\sqrt{2}}\bar{\chi}^m\frac{\Box}{\partial_-}\chi_m. \quad (2.48)$$

Next we turn to the superspace interaction terms. As explained in [14], these terms should be derivable by utilizing the non-linearly realized Lorentz and supersymmetry transformations (and the irreducibility of the light-cone superfield). However, this will not be done here for reasons that will become clear in the following. Instead we note that in the interaction terms the superfield Φ appears only to fourth and sixth power due to the structure of the component action. For instance, the terms involving only the field C_{mn} are

$$\begin{aligned} \mathcal{L}|_C = & \frac{1}{4}C_{mn}^A\Box\bar{C}^{Amn} - \frac{1}{4}(C_{mn}^A\partial_-\bar{C}^{Bmn})\frac{1}{\partial_-}(C_{pq}^C\partial_-\bar{C}^{Dpq})f^{ABCD} \\ & - \frac{1}{16}\frac{1}{\partial_-}(C_{mn}^C\partial_-\bar{C}^{Dmn})(C_{pq}^A\bar{C}^{Bpq})\frac{1}{\partial_-}(C_{rs}^G\partial_-\bar{C}^{Hrs})f^{CDAE}f^{GHBE} \\ & - \frac{1}{96}(C_{mn}^A\bar{C}^{Emn})(C_{pq}^B\bar{C}^{Fpq})(C_{rs}^C\bar{C}^{Grs})f^{ABCD}f^{EFGD}. \end{aligned} \quad (2.49)$$

Note that $C_{mn}^A\bar{C}^{Bmn}$ is symmetric in A and B due to the duality constraint, while if a derivative is inserted between the two fields it also has an antisymmetric piece.

Some interesting features of these pure C terms emerge if we try to express them in superspace. First, the quartic C term seems to tell us that the corresponding superspace term is just

$$S_4 = -2^{-7} \int d^3xd^4\theta d^4\bar{\theta} \frac{1}{16} \left((\Phi^A\partial\Phi^B)\frac{1}{\partial_-}(\bar{\Phi}^C\partial_-\bar{\Phi}^D)f^{ABCD} + c.c. \right) \quad (2.50)$$

due to the following facts: Φ^A has dimension $+1/2$, the whole $\mathcal{N} = 8$ superspace measure has dimension -1 , and C is not accompanied by any derivatives in the superfield.

Secondly, the sixth order terms in Φ^A must also contain some ∂_- derivatives and/or covariant derivatives d_m (and its complex conjugate). Terms in the superspace Lagrangian can be constructed for any set (of total dimension -2) of these derivatives and it may be that a combination of such terms is needed. However, since the last C term above has no ∂_- at all⁵, it is hard to see how any option with explicit ∂_- 's could be realized. Thus using explicit covariant derivatives d_m (and its complex conjugate) seems to be the only possibility. Note that this issue does not arise for $\mathcal{N} = 4$ super-Yang-Mills in four dimensions where the corresponding term is of order four in the complex superfield.

⁵Note that no such term exists constructed from only A 's and \bar{A} 's.

Another option is to consider the related $\mathcal{N} = 6$ ABJM theories [8]. These might in fact have a more natural light-cone superspace formulation based on a complex dimensionless superfield (and its conjugate) whose first component is the fermionic field divided by ∂_- . Note that for $\mathcal{N} = 6$ the total superspace measure is dimensionless just as in $\mathcal{N} = 4$ SYM in four dimensions. This could, e.g., mean that one can do without explicit d_m 's in the construction of the superspace Lagrangian also in this case.

3. The topologically gauged BLG theory on the light-cone

In this section we first review the $\mathcal{N} = 8$ conformal supergravity theory discussed recently in [1]. As noticed already in [17], the Lagrangian consists of three Chern-Simons-like terms, one for each one of the fields in the on-shell theory, the dreibein, the Rarita-Schwinger, and the R-symmetry gauge field. The number of derivatives in the corresponding 'Chern-Simons' terms are three, two and one, respectively. In the second part of this section these terms are analyzed in the light-cone gauge which will make the absence of physical degrees of freedom clear. It will also show how the derivative structure can be compatible with superconformal invariance in the matter sector.

3.1 Review

The off-shell field content of three-dimensional $\mathcal{N} = 8$ conformal supergravity is [25]

$$e_\mu{}^\alpha [0], \chi_\mu^i [-1/2], B_\mu^{ij} [-1], b_{ijkl} [-1], \rho_{ijk} [-3/2], c_{ijkl} [-2], \quad (3.1)$$

where the conformal dimensions are given in the square brackets⁶. It is possible to construct an on-shell topological Lagrangian from a set of Chern-Simons terms [1] (see also [17]) using only the three gauge fields of 'spin' 2, 3/2 and 1, i.e., $e_\mu{}^\alpha [0]$, $\chi_\mu^i [-1/2]$, $B_\mu^{ij} [-1]$.

The Lagrangian is a generalization of the $\mathcal{N} = 1$ case derived in [26] (see also [16]) and takes the form

$$L = \frac{1}{2} \epsilon^{\mu\nu\rho} \text{Tr}_\alpha (\tilde{\omega}_\mu \partial_\nu \tilde{\omega}_\rho + \frac{2}{3} \tilde{\omega}_\mu \tilde{\omega}_\nu \tilde{\omega}_\rho) - \epsilon^{\mu\nu\rho} \text{Tr}_i (B_\mu \partial_\nu B_\rho + \frac{2}{3} B_\mu B_\nu B_\rho) - i e^{-1} \epsilon^{\alpha\mu\nu} \epsilon^{\beta\rho\sigma} (\tilde{D}_\mu \tilde{\chi}_\nu \gamma_\beta \gamma_\alpha \tilde{D}_\rho \chi_\sigma), \quad (3.2)$$

where $\tilde{\omega}$ is the spin connection and the traces in the first and second terms are over the vector representation of the Lorentz group $SO(1,2)$ and the R-symmetry group $SO(8)$, represented by indices α and i , respectively⁷.

⁶Here the index i can be any of the three eight-dimensional representations of $SO(8)$.

⁷To conform with the original work reviewed in this section we keep the notation i although the index for an $SO(8)$ vector representation was denoted I in the previous section.

We will frequently use the notation [26]

$$f^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{D}_\nu\chi_\rho, \quad (3.3)$$

which makes the Rarita-Schwinger term read

$$-4i\bar{f}^\mu\gamma_\beta\gamma_\alpha f^\nu(e_\mu{}^\alpha e_\nu{}^\beta e^{-1}), \quad (3.4)$$

where we have spelt out explicitly all dependence of the dreibein that needs to be varied when checking supersymmetry.

The standard procedure to obtain local supersymmetry is to start by adding Rarita-Schwinger terms to the dreibein-compatible ω in order to obtain a supercovariant version of it. That is, we define

$$\tilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + K_{\mu\alpha\beta}, \quad (3.5)$$

where

$$\omega_{\mu\alpha\beta} = \frac{1}{2}(\Omega_{\mu\alpha\beta} - \Omega_{\alpha\beta\mu} + \Omega_{\beta\mu\alpha}), \quad (3.6)$$

with

$$\Omega_{\mu\nu\alpha} = \partial_\mu e_\nu{}^\alpha - \partial_\nu e_\mu{}^\alpha, \quad (3.7)$$

and contorsion given by

$$K_{\mu\alpha\beta} = -\frac{i}{2}(\chi_\mu\gamma_\beta\chi_\alpha - \chi_\mu\gamma_\alpha\chi_\beta - \chi_\alpha\gamma_\mu\chi_\beta). \quad (3.8)$$

This combination of spin connection and contorsion is supercovariant, i.e., derivatives on the supersymmetry parameter cancel out if $\tilde{\omega}_{\mu\alpha\beta}$ is varied under the ordinary transformations of the dreibein and Rarita-Schwinger field:

$$\delta e_\mu{}^\alpha = i\bar{\epsilon}\gamma^\alpha\chi_\mu, \quad \delta\chi_\mu = \tilde{D}_\mu\epsilon. \quad (3.9)$$

The covariant derivative appearing in the Lagrangian and in the variation of the Rarita-Schwinger field takes the following form acting on a spinor

$$\tilde{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon + \frac{1}{4}B_{\mu ij}\Gamma^{ij}\epsilon, \quad (3.10)$$

that is, both the Lorentz $SO(1,2)$ and the R-symmetry $SO(8)$ groups are gauged.

As explicitly demonstrated in [1] the above Lagrangian is $\mathcal{N} = 8$ supersymmetric (up to a total divergence) under the above transformations of the dreibein and the Rarita-Schwinger field together with a transformation of the $SO(8)$ R-symmetry gauge field $B_{\mu ij}$ that will be determined in the course of the calculation. This superconformal $\mathcal{N} = 8$ supergravity theory can then be coupled to the BLG theory as also discussed in [1].

It is convenient to introduce the dual $SO(8)$ R-symmetry and and curvature fields (see [26])

$$G_{ij}^{*\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}G_{\nu\rho ij}, \quad \tilde{R}^{*\mu}_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{R}_{\nu\rho\alpha\beta} \quad (3.11)$$

and similarly for $\tilde{\omega}$, as well as the double and triple duals

$$\tilde{R}^{**\mu,\alpha} = \frac{1}{2}\epsilon^{\alpha\beta\gamma}\tilde{R}^{*\mu}_{\beta\gamma}, \quad \tilde{R}^{***}_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\alpha}\tilde{R}^{**\nu,\alpha}, \quad (3.12)$$

where in the last expression only the contorsion part of the Riemann tensor contributes. In fact, one can show that

$$\tilde{R}^{***}_{\mu} = i\bar{\chi}_{\nu}\gamma_{\mu}f^{\nu}. \quad (3.13)$$

One also finds that that

$$\delta\tilde{\omega}_{\mu}^{*\alpha} = -2i(\bar{\epsilon}\gamma_{\mu}f^{\alpha} - \frac{1}{2}e_{\mu}^{\alpha}\bar{\epsilon}\gamma_{\nu}f^{\nu}). \quad (3.14)$$

Combining this result with the fact that the commutator of two supercovariant derivatives, acting on a spinor, is

$$[\tilde{D}_{\mu}, \tilde{D}_{\nu}] = \frac{1}{4}\tilde{R}_{\mu\nu\alpha\beta}\gamma^{\alpha\beta} + \frac{1}{4}G_{\mu\nu ij}\Gamma^{ij}, \quad (3.15)$$

we find that the symmetric part of $R^{**\mu,\alpha}$ cancels in the supersymmetry variation of the dreibein and gravitino Chern-Simons terms. Performing also the variation of the Chern-Simons term for the $SO(8)$ gauge field we find that also $G_{ij}^{*\mu}$ cancels provided we choose the variation of $B_{\mu ij}$ to be

$$\delta B_{\mu}^{ij} = -\frac{i}{2}\bar{\epsilon}\Gamma^{ij}\gamma_{\nu}\gamma_{\mu}f^{\nu}. \quad (3.16)$$

Inserting these variations into δL gives

$$\begin{aligned} \delta L &= \delta L_1 + \delta L_2 + \delta L_3 + \delta L_4, \\ \delta L_1 &= 4\bar{\epsilon}(\gamma_{\alpha}\gamma_{\beta}f^{\alpha})\bar{f}^{\mu}\gamma^{\beta}\chi_{\mu}, \\ \delta L_2 &= 8\bar{f}^{\mu}(\gamma_{\alpha}\gamma_{\beta}f^{\alpha})(\bar{\epsilon}\gamma^{\beta}\chi_{\mu} - \frac{1}{2}e_{\mu}^{\beta}\bar{\epsilon}\gamma^{\nu}\chi_{\nu}), \\ \delta L_3 &= 4(\bar{f}^{\alpha}\gamma_{\beta}\gamma_{\alpha})\gamma_{\gamma}\chi_{\mu}\epsilon^{\beta\mu\nu}(\bar{\epsilon}\gamma_{\nu}f^{\gamma} - \frac{1}{2}e_{\nu}^{\gamma}\bar{\epsilon}\gamma^{\rho}f_{\rho}), \\ \delta L_4 &= -\frac{1}{2}(\bar{f}^{\alpha}\gamma_{\beta}\gamma_{\alpha})\Gamma^{ij}\chi_{\mu}\epsilon^{\mu\beta\gamma}\bar{\epsilon}\Gamma_{ij}(\gamma_{\delta}\gamma_{\gamma}f^{\delta}). \end{aligned} \quad (3.17)$$

In order to show that the variation of the Lagrangian vanishes some of the terms in the above expression must be rearranged by Fierz transformations. By applying the Fierz transformations to δL_1 and δL_3 above and expressing all terms so obtained in the Fierz basis one can show, after some $\mathcal{N} = 1$ Fierz calculations, that they exactly cancel δL_2 . This is the result of Deser and Kay [26].

It now becomes rather easy to establish that also for $\mathcal{N} = 8$ the variation will vanish when δL_4 is included and use is made of the full $\mathcal{N} = 8$ Fierz identity for $SO(8)$ spinors of the same chirality, i.e.,

$$\begin{aligned}\bar{A}B\bar{C}D = & -\frac{1}{16}(\bar{A}D\bar{C}B + \bar{A}\gamma_\alpha D\bar{C}\gamma_\alpha B \\ & -\frac{1}{2}\bar{A}\Gamma^{ij}D\bar{C}\Gamma^{ij}B - \frac{1}{2}\bar{A}\gamma_\alpha\Gamma^{ij}D\bar{C}\gamma_\alpha\Gamma^{ij}B \\ & +\frac{1}{48}\bar{A}\Gamma^{ijkl}D\bar{C}\Gamma^{ijkl}B + \frac{1}{48}\bar{A}\gamma_\alpha\Gamma^{ijkl}D\bar{C}\gamma_\alpha\Gamma^{ijkl}B).\end{aligned}\quad (3.18)$$

This theory is also locally scale invariant (denoted by an index Δ) and possesses $\mathcal{N} = 8$ superconformal (shift) symmetry (denoted by S) with the following transformation rules (where ϕ is the local scale parameter and η the local shift parameter)

$$\begin{aligned}\delta_\Delta e_\mu^\alpha &= -\phi(x)e_\mu^\alpha, \\ \delta_\Delta \chi_\mu &= -\frac{1}{2}\phi(x)\chi_\mu, \\ \delta_\Delta B_\mu^{ij} &= 0,\end{aligned}\quad (3.19)$$

and

$$\begin{aligned}\delta_S e_\mu^\alpha &= 0, \\ \delta_S \chi_\mu &= \gamma_\mu \eta, \\ \delta_S B_\mu^{ij} &= \frac{i}{2}\bar{\eta}\Gamma^{ij}\chi_\mu.\end{aligned}\quad (3.20)$$

3.2 Light-cone analysis

In this subsection we will analyze the three Chern-Simons terms that make up the $\mathcal{N} = 8$ superconformal supergravity theory reviewed above. In [1] the superconformal supergravity theory was coupled to the ordinary (ungauged) BLG theory under the assumption that the supergravity sector does not add any new propagating degrees of freedom. This is, in fact, quite a natural property to expect from a theory that consists of just Chern-Simons terms. However, in the case of the $\mathcal{N} = 8$ superconformal supergravity theory only one of the three Chern-Simons terms is a conventional one-derivative term, namely the one for the gauged $SO(8)$ R-symmetry. The other two terms, on the other hand, are unusual due to their number of derivatives, three and two for the gravitational and the Rarita-Schwinger one, respectively. Despite these complications, in [27] the Chern-Simons term constructed from the metric compatible spin connection was rewritten as an $SO(2,3)$ Chern-Simons theory making clear some of its topological properties.

Here we will instead use light-cone techniques and analyze all three terms in the $\mathcal{N} = 8$ superconformal supergravity theory in a similar fashion. The light-cone treatment of the Chern-Simons term for the gauged $SO(8)$ R-symmetry is of course exactly the same as for the BLG vector field $\tilde{A}_{\mu b}^a$ discussed in section two.

Before going into the details of the light-cone analysis of the other two fields in the supergravity theory we note that also they have two field components each off-shell after making use of the superconformal gauge invariance. Although the two Chern-Simons terms are much more complicated one might hope that also here these two field components will be completely determined. As we will see below, this is indeed the case but the situation is slightly more interesting than that.

Next we turn to the Rarita-Schwinger (or gravitino) term

$$L = -ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu\bar{\chi}_\nu\gamma_\beta\gamma_\alpha\tilde{D}_\rho\chi_\sigma) + e\bar{\chi}_\mu J^\mu, \quad (3.21)$$

where we added a coupling term to an unspecified supercurrent. At linear order in the Rarita-Schwinger field χ_μ the field equation reads

$$2i\epsilon^{\mu\alpha\nu}\epsilon^{\beta\rho\sigma}\gamma_\beta\gamma_\alpha\partial_\nu\partial_\rho\chi_\sigma = -J^\mu, \quad (3.22)$$

which looks rather non-standard.

To start with we use the Q supersymmetry transformation rule, $\delta\chi_\mu = \partial_\mu\epsilon$ to set $\chi_- = 0$. Then we would like to go on and use the S superconformal transformations $\delta\chi_\mu = \gamma_\mu\eta$ to set one of the remaining two field components of χ_μ to zero. However, this is in direct conflict with $\chi_- = 0$ since χ_- will be affected by the superconformal transformation. This means that we have to design a new superconformal transformation S' that does not have this problem. We define

$$S'(\eta) = S(\eta) + Q(\epsilon = -\frac{\gamma_-}{\partial_-}\eta) = \gamma_\mu\eta - \partial_\mu(\frac{\gamma_-}{\partial_-}\eta), \quad (3.23)$$

which satisfies $\delta'_S\chi_- = \gamma_-\eta + \partial_-(-\frac{\gamma_-}{\partial_-}\eta) = 0$. We also find that

$$\begin{aligned} \delta'_S\chi_+ &= \gamma_+\eta - (\frac{\gamma_-}{\partial_-}\partial_+\eta) \\ \delta'_S\chi &= \gamma\eta - (\frac{\gamma_-}{\partial_-}\partial\eta). \end{aligned} \quad (3.24)$$

Thus we see that the P^- projected parts of χ_+ and χ can both be set to zero, using the first and second spin component of η , respectively⁸. So the gauge choices we will use are

$$\chi_- = P^-\chi_+ = P^-\chi = 0. \quad (3.25)$$

Next we would like to implement these conditions in the field equation and analyze the resulting equations. The covariant linearized equations are

$$\partial^\alpha\partial_\alpha\chi_\mu - \partial_\mu(\partial^\nu\chi_\nu) + \epsilon_\mu{}^{\nu\rho}(\gamma^\alpha\partial_\alpha)\partial_\nu\chi_\rho = -iJ_\mu. \quad (3.26)$$

Using the gauge conditions inside the two expressions in parenthesis these equations become

$$\partial^\alpha\partial_\alpha\chi_\mu - \partial_\mu(-\partial_-\chi_+ + \partial\chi) + \epsilon_\mu{}^{\nu\rho}(\gamma^-\partial_- + \gamma\partial)\partial_\nu\chi_\rho = -iJ_\mu. \quad (3.27)$$

⁸This is of course restricted to $p^+ \neq 0$, and for χ in addition to $p^+ \neq -\sqrt{2}p$.

Then the $\mu = -$ component of this equation reads

$$-\partial_-(-\partial_- \chi_+ + \partial \chi) - (\gamma^- \partial_- + \gamma \partial)(\partial_+ \chi - \partial \chi_+) = -iJ_-, \quad (3.28)$$

which if further projected with P^+ and P^- gives the two equations

$$\partial_-^2 \chi_+ - \partial_- \partial \chi + -\gamma \partial \partial_- \chi = -iP^+ J_-, \quad (3.29)$$

$$\partial_-^2 \chi = \frac{i}{2} \gamma^+ P^- J_-. \quad (3.30)$$

Similarly, the $\mu = +$ component is

$$\partial^\alpha \partial_\alpha \chi_+ - \partial_+(-\partial_- \chi_+ + \partial \chi) + (\gamma^- \partial_- + \gamma \partial)(\partial_+ \chi - \partial \chi_+) = -iJ_+, \quad (3.31)$$

which splits into the two equations

$$\begin{aligned} \partial^\alpha \partial_\alpha \chi_+ + \partial_+ \partial_- \chi_+ - \partial_+ \partial \chi + \gamma \partial(\partial_+ \chi - \partial \chi_+) &= -iP^+ J_+, \\ \gamma^- \partial_- (\partial_+ \chi - \partial \chi_+) &= -iP^- J_+. \end{aligned} \quad (3.32)$$

Finally, the $\mu = 2$ component reads (dropping the index 2)

$$\partial^\alpha \partial_\alpha \chi - \partial(-\partial_- \chi_+ + \partial \chi) + (\gamma^- \partial_- + \gamma \partial) \partial_- \chi_+ = -iJ_\mu, \quad (3.33)$$

and its two component equations are

$$\begin{aligned} \partial_-^2 \chi_+ &= -\frac{i}{2} P^- J, \\ \partial^\alpha \partial_\alpha \chi + \partial \partial_- \chi_+ - \partial^2 \chi + \gamma \partial \partial_- \chi_+ &= -iP^+ J. \end{aligned} \quad (3.34)$$

So, we find immediately that the two component fields that remain after gauge fixing are determined by two of the above equations:

$$\begin{aligned} \chi_+ &= -\frac{i}{2\partial_-^2} \gamma^+ P^- J, \\ \chi &= -\frac{i}{2\partial_-^2} \gamma^+ P^- J_-. \end{aligned} \quad (3.35)$$

Furthermore, we find from the other component equations conditions also on $\partial_+ \chi$ and $\partial_+ \chi_+$, namely

$$\begin{aligned} \partial_+ \chi_+ &= -\frac{i}{2\partial_-} P^+ J^+, \\ \partial_+ \chi &= -\frac{i}{2\partial_-^2} \gamma^+ P^- J_-. \end{aligned} \quad (3.36)$$

The last two component equations are just the restrictions on the supercurrent needed to make it compatible with both local supersymmetry and local superconformal symmetry. Also the fact that both field components and their first ∂_+ derivative are determined lead to constraints on the supercurrent. That these conditions are

satisfied by the supercurrents in the topologically gauged BLG can easily be checked. One explicit example of this will be presented at the end of the section.

Finally, we turn to the Chern-Simons term for the metric compatible spin connection. Also here we have two symmetries to take into account, the reparameterization invariance and local scale invariance. As usual, the theory is also locally Lorentz invariant so we first use this fact to put $\omega_{-\alpha\beta} = 0$. This condition follows also if we use the Lorentz and coordinate invariances to impose on the dreibein the following constraints at the linearized level:

$$e_-^- = 1, \quad e_-^+ = 0, \quad e_-^2 = 0, \quad (3.37)$$

and that it is symmetric. Of the remaining dreibein components only e_2^2 is affected by a (linear) local rescaling (after a light-cone redefinition making it orthogonal to the previous gauge choices) and thus we can also set

$$e_2^2 = 0. \quad (3.38)$$

The spin two Lagrangian is (keeping a spin connection with torsion)

$$L = \frac{1}{2}\epsilon^{\mu\nu\rho}Tr_\alpha(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho) + ee_\mu{}^\alpha T^\mu{}_\alpha. \quad (3.39)$$

To find the field equations we first vary with respect to the spin connection which gives

$$\delta L = -\frac{1}{2}\epsilon^{\mu\nu\rho}\delta\tilde{\omega}_{\mu\alpha\beta}\tilde{R}_{\nu\rho}{}^{\alpha\beta} = 2\delta\tilde{\omega}_{\mu\alpha}^*\tilde{R}^{**\mu\alpha}. \quad (3.40)$$

Then we need also the dreibein variation of the spin connection which reads, setting the torsion part to zero,

$$\delta\omega_{\mu\alpha\beta} = e_{[\alpha}{}^\sigma D_{|\mu}\delta e_{\sigma|\beta]} - e_{[\alpha}{}^\sigma D_{|\sigma}\delta e_{\mu|\beta]} + e_{[\alpha}{}^\sigma e_{\beta]}{}^\nu e_\mu{}^\gamma D_{[\nu}\delta e_{\sigma]\gamma}. \quad (3.41)$$

The field equation without torsion is then found to be (see, e.g. [27])

$$D_\rho W_{\mu\alpha} - D_\mu W_{\rho\alpha} = -\frac{1}{2}\epsilon_{\rho\mu\sigma}T^\sigma{}_\alpha, \quad (3.42)$$

where the LHS is related to the Cotton tensor $C_{\mu\nu} = \epsilon_\mu{}^{\rho\sigma}D_\rho W_{\sigma\nu}$ and

$$W_{\mu\alpha} = R_{\mu\alpha} - \frac{1}{4}e_{\mu\alpha}R. \quad (3.43)$$

However, we will need only its linearized version:

$$\begin{aligned} & -\partial^\alpha\partial_\alpha\partial_{[\rho}h_{\mu]\nu} + \partial_\nu\partial^\sigma\partial_{[\rho}h_{\mu]\sigma} \\ & -\frac{1}{2}\eta_{\nu[\rho}(\partial^\alpha\partial_\alpha\partial_{\mu]}h_{\sigma}{}^\sigma - \partial_{\mu]}\partial^\alpha\partial^\beta h_{\alpha\beta}) \\ & = -\frac{1}{2}\epsilon_{\rho\mu\sigma}T^\sigma{}_\nu. \end{aligned} \quad (3.44)$$

The next step is to implement the above set of gauge conditions on the linearized dreibein $h_{\mu\nu}$. From now on we denote the remaining fields (up to a factor 2) by h_{++} and h_{+2} since they are really metric components. With these gauge choices we also find that the trace of h vanishes, and that the field equations simplify to

$$\begin{aligned} & -2\partial^\alpha\partial_\alpha\partial_{[\rho}h_{\mu]\nu} - 2\partial_-\partial^\nu\partial_{[\rho}h_{\mu] +} + 2\partial_\nu\partial^2\partial_{[\rho}h_{\mu]2} \\ & + \eta_{\nu[\rho}\partial_{\mu]}(\partial_-^2h_{++} - 2\partial_-\partial_2h_{+2}) \\ & = -\epsilon_{\rho\mu\sigma}T^\sigma{}_\nu. \end{aligned} \quad (3.45)$$

Reading off the equations for each possible index combination we find nine equations which imply that the stress tensor is symmetric and traceless. The remaining five equations can be solved for h_{++} and h_{+2} and ∂_+ derivatives on them in terms of some expressions involving only the stress tensor, thus giving a picture similar to the one obtained above for the Rarita-Schwinger field. Explicitly we find

$$\begin{aligned} \partial_-^3h_{++} &= -2T_{2-}, \\ \partial_-^3h_{+2} &= -T_{--}, \\ \partial_-^2(\partial_+h_{+2}) - \frac{1}{2}\partial_-^2\partial_2h_{++} &= \frac{1}{2}T_{22}, \\ \partial_-^2(\partial_+h_{++}) - 2\partial_-\partial_2^2h_{++} + 4\partial_-(\partial_+h_{+2}) &= 2T_{2+}, \\ \partial_-(\partial_+^2h_{+2}) - \partial_2^2(\partial_+h_{+2}) + \partial_2^2(\partial_+h_{+2}) &= -T_{++}. \end{aligned} \quad (3.46)$$

The solution is

$$\begin{aligned} h_{++} &= -2\partial_-^{-3}T_{2-}, \\ h_{+2} &= -\partial_-^{-3}T_{--}, \\ \partial_+h_{++} &= 2\partial_-^{-2}T_{2+} - 2\partial_-^{-3}(\partial_2T_{22}), \\ \partial_+h_{+2} &= \frac{1}{2}\partial_-^{-2}T_{22} - \partial_-^{-3}(\partial_2T_{2-}), \\ \partial_+^2h_{+2} &= -\partial_-^{-1}T_{++} + \partial_-^{-3}(\partial_2^2T_{22}). \end{aligned} \quad (3.47)$$

Finally, as promised, we check that the equations above involving extra ∂_+ 's are consistent with expected properties of the stress tensor. Consider for instance the stress tensor for a free scalar field. The locally scale invariant action for a scalar field (in three dimensions) is

$$L = -\frac{1}{2}e(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{R}{8}\phi^2). \quad (3.48)$$

This Lagrangian leads to a stress tensor that is traceless and satisfies the ∂_+ conditions mentioned above. The curvature term turning the free scalar into a conformal theory is present also in the topologically gauged BLG derived in [1].

4. Conclusions and comments

Light-cone techniques are often used to keep track of the physical propagating degrees of freedom while making it possible to solve for, and thus effectively eliminate,

all unphysical local modes. In this paper this is done for the three-dimensional superconformal BLG theory describing a pair of M2 branes. The result is an action expressed completely in terms of the physical light-cone modes A, C_{mn}, χ_m . We also take a first step towards a light-cone superspace formulation in the spirit of previous work on $\mathcal{N} = 4$ SYM in four dimensions [13, 14, 15] by giving the kinetic term and a possible answer for the quartic term in superspace. Some issues related to the sixth order terms are also discussed.

We then apply these methods to the topologically gauged BLG theory, discussed recently in [1], in order to find out exactly how the physical modes are embedded into such a topological higher derivative theory. In spite of the many derivatives it is possible to solve for all components of the gauge fields (the dreibein, Rarita-Schwinger and R-symmetry gauge potential) in terms of BLG matter fields here represented by general currents. However, one also finds expressions which provide relations between ∂_+ derivatives acting on these field components and the currents. These latter equations generate constraints on the currents which are shown to be satisfied in the simple example of a conformally coupled scalar field.

We are in this paper dealing with two theories based on Chern-Simons terms, in the BLG case for ordinary Yang-Mills fields and in the gauged case also for the gravitational fields in $\mathcal{N} = 8$ superconformal gravity, none of which have any propagating physical degrees of freedom. Therefore, it would be interesting to study the global modes of these topological theories. Results in this direction are well-known in the ungauged cases, and in particular for ABJM theories (see, e.g., [28] and references therein). An analysis of the global modes is easier done using methods not related to the light-cone gauge, but it may not be impossible (e.g., a continuation to Euclidean signature can be performed as explained in [14]). In fact, it would be interesting to know to what extent, if at all, global modes can be studied in the light-cone gauge. Topological aspects of the gravitational Chern-Simons theory have also been discussed some time ago in [27] using a reformulation in terms of an $SO(3, 2)$ ordinary Chern-Simons theory.

The resulting light-cone Lagrangian obtained in this paper rely on a proper definition of the inverse operator $(\partial_-)^{-1}$ (see, e.g., [14]) and avoids the need for higher powers of it. Such higher powers do, however, appear in light-cone discussions of $\mathcal{N} = 8$ four-dimensional gravity [15] and may be defined by repeated use of the definition for a single inverse operator. If the results obtained here have any bearing on the more complicated gravity theories remains to be seen.

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